REVIEW OF THE FRENCH EDITION OF HALSTED'S "RATIONAL GEOMETRY."

By JOHN A. EIESLAND, University of West Virginia.

Rational Geometry. By George Bruce Halsted. Second Edition. New York and London: John Wiley and Sons. 1907. Pp. viii+274.

Dr. George-Bruce Halsted. *Géométrie Rationnelle. Traité Elémentaire de la Science de la l'Espace.* Traduction Française par Paul Barbarin. Avec une Préface de C.-A. Laisant. Paris, Gauthier-Villars. 1911. Pp. iv+296.

The French translation by Professor Barbarin of the second edition of Professor Halsted's *Rational Geometry* is a handsome volume published by the book firm of Gauthier-Villars. The book is prefaced by Professor C. A. Laisant, who is well known to American students of mathematics. It is now seven years since the publication of the first edition of this work, a review of which by Professor Davisson of Indiana University was published in the *Bulletin of the American Mathematical Society*, Vol. XI, pp. 330-336.

Poincaré has characterized Halsted's introduction of Hilbert's ideas and principles in an elementary text-book as follows:

"Introduire ce principe dans l'enseignement, c'est bien pour le coup rompre les ponts avec l'intuition sensible, et c'est là je l'avoue, une hardiesse qui me parait presque une témérité."

Daring and rash it may be, but Professor Halsted certainly deserves credit for his temerity, for he has given to American teachers and students of mathematics a well-written text-book on Geometry in which the fruitful ideas and principles of Hilbert have been most ably interpreted and applied. To what extent the book can be used in elementary instruction, as a high-school text-book, for instance, the writer is not prepared to discuss, but it seems reasonable to suppose that it can be taught with the best results to students who have had a preliminary course in geometry such as the one published some time ago by E. Borel,* in which intuitional methods are largely used and formal demonstrations relegated to a second place. The prevalent American custom of introducing a student to formal demonstrative geometry without any preceding intuitional geometry seems pedagogically unwise and is largely responsible for the distaste of the average high-school student for mathematical subjects. The century-long authority of Euclid will of course prevent the book from becoming popular or a good seller. To the average teacher of mathematics with limited training and knowledge of modern mathematics it will seem revolutionary and strange because it is new, not because it is inherently more difficult than Euclid or any of his modern followers. In some respects it is simpler. With Euclid

* E. Borel, *Géométrie, Premier et second cycles.* Paris, Armand Colin, 1905.

the theory of proportion involves the difficult idea of limits and continuity (commensurability); here we have a sect-calculus (Ch. VII) based on very simple assumptions and on Schur's very elementary proof of Pascal's theorem. The vexing question about definitions of point, straight line, plane, is solved by not defining them at all, but assuming them as the "elements" or "things" that form the material of the structure.*

It is not our purpose to review each particular chapter in detail. The changes which have been made in this second and revised edition are all decided improvements. The sect-calculus is introduced by Schur's proof of Pascal's theorem on the basis of which the commutative, associative and distributative laws for addition (subtraction), multiplication, and division of sects are proved. This seems preferable to the method of the first edition, which made use of the circle.

The author tries to avoid as much as possible the use of the Archimedes assumption. In the first edition he was disposed to eliminate it altogether, being satisfied with a mere statement of it. However, in Ch. XI on "Length and Content of the Circle" it was found impossible to avoid the use of this assumption and — a consequence of it — the assumption of a unique sect equal to the length of the circle. From the standpoint of rigor this seems satisfactory, although it is not made quite clear in the text how the Archimedes assumption is involved in the assumption: "There is one and only one sect greater than the perimeter of any inscribed polygon and less than the perimeter of any circumscribed polygon, namely, the length of the circle." The real content of the Archimedes assumption, in so far as it applies to this postulate, is thus left as an unexplained mystery. Perhaps this is the only way out of the dilemma; to go into details might also necessitate saying something about the "axiom of completeness" and this would have been decidedly beyond the scope of an elementary book. On the whole we think the author has skilfully avoided the difficulties of this part of the subject without sacrificing rigor.

In writing an elementary text-book based on Hilbert's system of axioms a difficulty presents itself that can only be conquered by evading it. Thus, in the first edition of Hilbert's *Foundations* we find an axiom II, 4 as follows: "Any four points on a straight line can always be so lettered, A, B, C, D, that B is between A and C and also between A and D, and furthermore C is between A and D and also between B and D." This assumption was later found to be a demonstrable theorem,† but we note that

* We note that the "National Committee of Fifteen on Geometric Syllabus" seems to take a similar view: "Certain concepts are so elementary that no simple terms exist by which to define them, although they can easily be explained. For example, *point, line, surface, space, angle, straight line, curve.* The committee recommends that teachers give more attention to instilling a clear concept of such terms and none to exact definitions." This is all very good indeed, but we find on page 30 of the same report among the general list of *postulates:* "A straight line is the shortest line between two points." First the student is told that a straight line is of illimitable extent, and then he is told to believe that such a line is the shortest line. This is truly what Max Dehn calls "Ein energischer widerspruch."

† Proved by R. L. Moore and E. H. Moore.

the proof is rather long and difficult for beginners and the author has therefore rightly placed it in appendix I. The student will readily grant the proposition, appeal being made to intuition, but we have here a striking illustration of the difficulty we meet when any appeal to intuition is denied, as it must be in a Rational Geometry.

And now quite recently in a paper published in the *Annalen*, B. 71, 1911, Rosenthal* has proved that the system of assumptions of congruence can be simplified; he shows that assumption III, 5† is wholly superfluous, the same being true of parts of III, 1 and III, 4. His proofs of these superfluous axioms are not very simple in an elementary sense, and in a future third edition of Halsted's geometry we shall perhaps have an appendix II devoted to these theorems, the proofs of which we hope the author will simplify sufficiently for elementary use. It follows also from Rosenthal's work that the five congruence assumptions for the geometry on the sphere (*Pure Spherics*, Ch. XVI, pp. 226-227, French edition) may be reduced by one and assumptions III, 1 and III, 4 simplified so as to correspond to the same group of assumptions for the plane.

In appendix II the author introduces the basic assumptions for the compasses. Here we find a notable improvement in the present edition, the two assumptions being reduced to one, viz.: "If a straight have a point within a circle, it has two points on the circle." Schur has pointed out that this assumption can be proved by means of the Archimedes assumption. The author wisely omits such proof, but he might have stated the fact, since the student easily gets an idea that this is an entirely new assumption independent of the rest.

A word may here be said about the nomenclature. Professor Halsted's innovations have not been generally accepted by text-book writers, but that proves nothing for or against his choice of terms. The word 'co-straight' will be objected to by purists as a hybrid; 'co-punctal' is better than 'concurrent,' if the idea of motion is to be eliminated; while the word 'sect' seems preferable to 'segment,' which is used to designate something else. The French translator has rendered 'sect' by 'segment,' 'co-straight' has for its equivalent 'collineaire,' and 'co-punctal' is translated 'coponctuelle.' We note with pleasure that the "Committee of Fifteen" has been kind enough to allow the use of the word "congruent." It may be that the author of *Rational Geometry* will live to see the day when a future more liberal "Committee of Fifteen" shall accept his nomenclature, if not *in toto*, yet the greater number of his terms.

As regards print, binding, and general appearance, the French edition is a real work of art. We shall say nothing about Professor Barbarin's translation; a competent judge, Professor Laisant, says: "Professor

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* *Vereinfachungen des Hilbertschen System der Kongruenzaxiome*, von Arthur Rosenthal in Munchen.

† Assumption III, 5 is: "If an angle (h, k) is congruent as well to the angle (h', k') as also to the angle (h'', k'') then is also the angle (h', k') congruent to the angle (h'', k'').

Halsted's book is lucidly written, the translation is at once faithful and smooth."

We congratulate Professor Halsted on the well-merited success that his book has attained. Sooner or later it will no doubt exercise a profound influence on the teaching of geometry in our country.

A METHOD FOR THE SOLUTION OF SIMULTANEOUS QUADRATIC EQUATIONS OF THE SYMMETRIC TYPE.

By B. E. MITCHELL, Nashville, Tennessee.

The following method is suggested but not developed in Fine's *College Algebra.*

The most general quadratic of the symmetric type between two variables is $a(x^2+y^2)+2bxy+c(x+y)+d=0$.

Let x and y be the roots of $u^2+pu+q=0$. Then

$$u=\begin{cases} x=\frac{1}{2}[-p\pm\sqrt{(p^2-4q)}] \\ y=\frac{1}{2}[-p\mp\sqrt{(p^2-4q)}] \end{cases}\ldots(\mathrm{I}).$$

The double sign is used in each because they are the roots of a symmetric equation.

From the theory of quadratic equations we have

$$x+y=-p\ldots(1),$$

and $$xy=q\ldots(2),$$

from which we calculate $$x^2+y^2=p^2-2q\ldots(3).$$

These are the three symmetric forms that occur in the general equation above, and are the only ones of the second degree.

Let us solve two simultaneous equations by this method.

$$\begin{cases} a_1(x^2+y^2)+2b_1xy+c_1(x+y)+d_1=0\ldots(1). \\ a_2(x^2+y^2)+2b_2xy+c_2(x+y)+d_2=0\ldots(2). \end{cases}$$

Then

$$\begin{cases} a_1p^2-2(a_1-b_1)q-c_1p+d_1=0\ldots(3). \\ a_2p^2-2(a_2-b_2)q-c_2p+d_2=0\ldots(4). \end{cases}$$

Each of the equations (3) and (4) is linear in q. Solving them for q, we have

$$2q=\frac{a_1p^2-c_1p+d_1}{a_1-b_1}=\frac{a_2p^2-c_2p+d_2}{a_2-b_2}\ldots(5).$$

Considering equation (5) we have the following:

The coefficient of p^2 is $a_2b_1-a_1b_2=-\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ which we call A.

The coefficient of p is $a_1c_1-a_2c_2-b_1c_2+b_2c_1=-\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 0 \end{vmatrix}$ which we call B.

The constant term is $a_2d_1-b_2d_1-a_1d_2+b_1d_2=\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ d_1 & d_2 & 0 \end{vmatrix}$ which we call C.

Hence we have $Ap^2+Bp+C=0$ yielding generally two values for p, and from (5) we may calculate the two corresponding values of q. We have then the two equations in u:

$$u^2+p_1u+q_1=0,$$
$$u^2+p_2u+q_2=0,$$

which give four pairs of values for x and y when substituted in (I).

In the application of the method to numerical problems it will usually be found easier to use it as a process rather than the above equations as formulae.

Let us apply the method to three numerical problems taken from Fine's *College Algebra*.

(1) $\begin{cases} x+y=5. \\ xy+36=0. \end{cases}$

Hence $p=-5$ and $q=-36$, so $u^2-5u-36=0$, and $u=\begin{cases} x=9 \text{ and } -4. \\ y=-4 \text{ and } 9. \end{cases}$

(2) $\begin{cases} x^2+xy+y^2=21. \\ x+\sqrt{(xy)}+y=7. \end{cases}$ $\begin{cases} p^2-2q+q=21. \\ -p+\sqrt{q}=7. \end{cases}$ $\begin{cases} p^2-q=21. \\ p^2-q+14p=-49. \end{cases}$

$p=-5$, $q=4$. $u^2-5u+4=0$. $(u-4)(u-1)=0$.

$u=\begin{cases} x=1 \text{ and } 4. \\ y=4 \text{ and } 1. \end{cases}$

(3) $\begin{cases} x^2+3xy+y^2+2x+2y=8. \\ 2x^2+2y^2+3x+3y=14. \end{cases}$

Then $p^2-2p+q=8$, and $2p^2-3p-2q=14$.

Eliminating q by addition, $4p^2-7p-30=0$, $p=-2$ and $\frac{15}{4}$, then $q=-0$ and $\frac{23}{16}$. So $u^2-2u=0\ldots(a)$, and $u^2+\frac{15}{4}u+\frac{23}{16}=0\ldots(b)$.

From (a), $u=\begin{cases} x=0 \text{ and } 2. \\ y=2 \text{ and } 0. \end{cases}$

From (b), $u=\begin{cases} x=\frac{1}{2}\left[-\frac{15}{4}\pm\sqrt{\left(\frac{133}{16}\right)}\right]. \\ y=\frac{1}{8}\left[-15\mp\sqrt{(133)}\right]. \end{cases}$

This method may also be used where the equations are symmetric with respect to x and $-y$. In this case we have

$$x-y=-p,\ xy=-q,\ \text{and}\ x^2+y^2=p^2-2q.$$

By calculating other symmetric functions the method can be used successfully in solving many equations of higher degree than the second.

SOME CONSTRUCTIONS LEADING TO CONICS.

By F. H. HODGE, Franklin College, Indiana.

Among the courses that find a place in collegiate mathematics one is usually given which involves the treatment of plane curves. Certain loci are described in such a course and their equations are derived and the properties indicated by the equations are discussed. It seems desirable to have a stock of plane loci which can be simply described, and the derivation of whose equations does not present great difficulty, but which are not treated in current text books. Such loci could be assigned to pupils as original exercises, and they would serve to stimulate the capacity for independent thought.

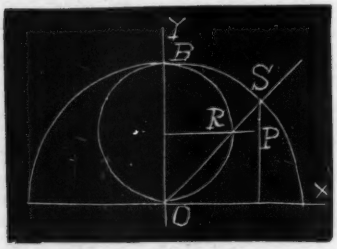

Fig. 1.

The four constructions which follow lead to conics. These are so simple and so directly analogous to a well known construction for the ellipse that it is scarcely possible that they are new though they are not found in current texts. The first two are described in detail, and the others merely suggested.

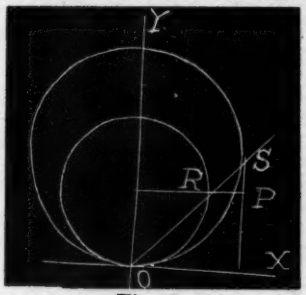

Fig. 2.

(1) Given two circles tangent internally at B and having the radius of the larger equal to the diameter of the smaller. Take the center of the larger circle as origin and the tangent to the smaller circle at that point as the x-axis, the y-axis being perpendicular to the x-axis. Draw a secant line through the origin, meeting the smaller circle at R and the larger circle at S. Through R draw a line parallel to the x-axis, and through S draw a line parallel to the y-axis. Call the point P in which these two lines meet. Required to find the locus of P as the secant line revolves about the origin as an axis. See Fig. 1.

(2) Here again are given two circles tangent internally, but no definite ratio is specified between the lengths of the radii. Take the point of tangency as the origin and the common tangent to the two circles as the x-axis, the y-axis being perpendicular to the x-axis. Draw the secant line as before and also the lines through R and S parallel to the x- and y-axes, respectively, and meeting at P. Required the locus of P as the secant line revolves. See Fig. 2.

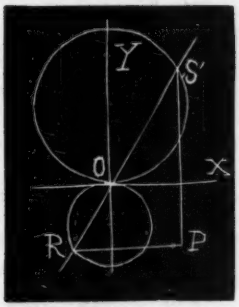

Fig. 3.

(3) This construction is similar to that of (2) except that the two circles are tangent externally, and the common tangent to the circles at the point of tangency is taken as the x-axis. See Fig. 3.

(4) Here the center of one circle lies on the circumference of the other circle. This center is taken as the origin, and the line joining it to the center of the other circle is taken as the y-axis. The secant and the other lines are drawn as in the other constructions, and the locus of P is required. See Fig. 4.

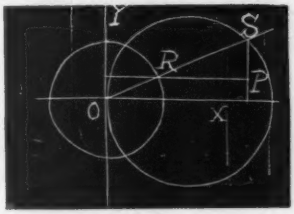

Fig. 4.

All four of these constructions can be shown by very elementary considerations to lead to conics, two of them giving ellipses, and the other two giving parabolas. Constructions similar to these can be made to yield an indefinite number of curves by simply replacing one or both of the circles by ellipses, parabolas, hyperbolas, or other curves. Such considerations, however, would lead us beyond the scope of an elementary course.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

368. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Michigan.

Solve the functional equation, $\dfrac{f(-x)}{f(x)}=r^{2x}$.

I. Solution by the PROPOSER.

Let $f(x)=A(x^2)+xB(x^2)$. Then the equation becomes

$$\frac{A(x^2)-xB(x^2)}{A(x^2)+xB(x^2)}=r^{2x};\ \text{whence}\ \frac{A(x^2)}{xB(x^2)}=\frac{1+r^{2x}}{1-r^{2x}}=-\frac{r^x+r^{-x}}{r^x-r^{-x}};$$

$$\text{or,}\ \frac{A(x^2)}{B(x^2)}=-\frac{r^x+r^{-x}}{(1/x)\ (r^x-r^{-x})}.$$

$\therefore A(x^2)=(r^x+r^{-x}).\phi(x^2),\ B(x^2)=-\frac{1}{x}(r^x-r^{-x}).\phi(x^2)$, where $\phi(x^2)$ is any function of x^2.

$\therefore f(x)=\frac{2}{r^x}.\phi(x^2)$.

II. Solution by S. LEFSCHETZ, Ph. D., The University of Nebraska.

The given functional equation can be written:

$$f(-x)r^{-x}=f(x)r^x.$$

Hence $f(x)r^x=\phi(x)$, where ϕ is any even function of x.

$\therefore f(x)=\frac{\phi(x)}{r^x}$.

Ex. $\phi(x)=\sum_0^\infty mA_m\cos mx$; $\phi(x)=\psi(x^2)$, etc.

Also solved by A. H. Holmes and J. Scheffer.

369. Proposed by WILLIAM HOOVER, Ph. D., Athens, Ohio.

If $f(m)=(1+x)^m$, and $f(n)=(1+x)^n$, why not obviously $f(m).f(n)=(1+x)^{m+n}=f(m+n)$?

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The proof given of $f(m)f(n)=f(m+n)$ is certainly incontestable; but it may also be proved directly thus: Differentiating with reference to m and n separately, we have

$$f(n)\ f'(m)=f'(m+n),\quad f'(n)\ f(m)=f'(m+n).$$

$\therefore f(n)\ f'(m)=f'(n)\ f(m)$; $\therefore \frac{f'(m)}{f(m)}=$ a constant $=a$ (say).

$\therefore f(m)=a^m$.

Solutions of 367 were received from A. M. Harding, Elmer Schuyler, S. Lefschetz, and the Proposer.

GEOMETRY.

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394. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

The joins of the excentres to the corresponding vertices of the pedal triangle are concurrent.

I. Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let $ABCD$ be the triangle of reference; p_1, the perpendicular AP_1 upon the side A, analogous notation making $P_1P_2P_3$ the pedal triangle. Using trilinear coördinates, the center C_1 of the escribed circle touching a is given by $(-a_1, a_1, a_1)$, and similarly for the other centers.

The point P_1 is given by $(0, p_1\cos C, p_1\cos B)$, and P_2, P_3 are set down by symmetry.

The equation to P_1C_1 is

$$\begin{vmatrix} a & \beta & \gamma \\ -a_1 & a_1 & a_1 \\ 0 & p_1\cos C & p_1\cos B \end{vmatrix} = \begin{vmatrix} a & a+\beta & a+\gamma \\ -1 & 0 & 0 \\ 0 & p_1\cos C & p_1\cos B \end{vmatrix}$$

$$= \begin{vmatrix} a+\beta & a+\gamma \\ \cos C & \cos B \end{vmatrix} = a(\cos B - \cos C) + \beta\cos B - \gamma\cos C = 0 \ldots (1).$$

The equations to P_2C_2, P_3C_2 are, by symmetry,

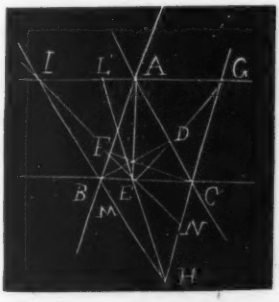
$$-a\cos A + \beta(\cos C - \cos A) + \gamma\cos C = 0 \ldots (2),$$
$$a\cos A - \beta\cos B + \gamma(\cos A - \cos B) = 0 \ldots (3).$$

(1), (2), and (3) meet in a point, since

$$\begin{vmatrix} \cos B - \cos C & \cos B & -\cos C \\ -\cos A & \cos C - \cos A & \cos C \\ \cos A & -\cos B & \cos A - \cos B \end{vmatrix}$$

$$= \begin{vmatrix} \cos B & \cos B & -\cos C \\ -\cos C - \cos A & \cos C - \cos A & \cos C \\ \cos B & -\cos B & \cos A - \cos B \end{vmatrix} = 0.$$

II. Solution by N. L. RORAY.



Given the triangle ABC; D, E, and F vertices of the pedal triangle; G, H, and I centers of the ex-circles; GM, HL, and IN sects joining ex-centers with the corresponding vertices of the pedal triangle.

To prove: GM, HL, and IN concurrent.

Proof. $\dfrac{\triangle HIL}{\triangle BIA} = \dfrac{IL.IH}{IA.IB}$, $\dfrac{\triangle BIA}{\triangle HBE} = \dfrac{IB.BA}{BH.BE}$

UofM

$$\therefore \frac{\triangle HIL}{\triangle HBE}=\frac{IL.IH.BA}{IA.BH.BE}=\frac{HI.HL}{HB.HE} \qquad \therefore \frac{IL}{IA}\cdot\frac{BA}{BE}=\frac{HL}{HE}.$$

Similarly, $\frac{GA}{GL}\cdot\frac{CE}{AC}=\frac{HE}{HL}$; $\frac{GN}{GC}\cdot\frac{AC}{AF}=\frac{IN}{IF}$; $\frac{HC}{HN}\cdot\frac{BF}{BC}=\frac{IF}{IN}$;

$$\frac{IB}{IM}\cdot\frac{AD}{AB}=\frac{GD}{GM}; \text{ and } \frac{HM}{HB}\cdot\frac{BC}{CD}=\frac{GM}{GD}.$$

$$\therefore \frac{IL.GN.HM}{GL.HN.IM}\cdot\frac{GA.IB.HC}{IA.BH.GC}\cdot\frac{CE.BF.AD}{BE.CD.AF}=1.$$

But by Ceva's Theorem, $\frac{CE.BF.AD}{BE.CD.AF}=1$.

Also, it is easily proved that $\frac{GA.IB.HC}{IA.BH.GC}=1$. $\therefore \frac{IL.GN.HM}{GL.HN.IM}=1$.

$\therefore GM$, HL, and IN are concurrent. Q. E. D.

397. Proposed by DAVID F. KELLEY, New York City.

If ABC be a semicircle and CD a perpendicular from C on the diameter AB, prove that the radius of the circle inscribed in the triangle ABC equals half the sum of the radius of the circle touching arc AC and the sides AD and DC of the triangle ADC, and the radius of the circle touching arc CB and sides DB and DC of triangle CDB, and that the centers of the three circles are collinear.

I. Solution by H. C. FEEMSTER, Professor of Mathematics, York College, York, Nebraska.

Let AB be the y-axis, CD the x-axis, r the radius of the given circle, $k=r-AD$.

Then $\begin{cases} x_1^2+(x_1+k)^2=(r-x_1)^2 \\ y_1=x_1, \end{cases}$ (1),

and $\begin{cases} x_{11}^2+(k-x_{11})^2=(r-x_{11})^2 \\ y_{11}=-x_{11}, \end{cases}$ (2),

are the relations that give the centers of the circles inscribed in ADC and BDC. The bisectors of the angles A and C intersect at

$$x=\frac{-2r+\sqrt{(2r^2-2rk)}+\sqrt{(2r^2+2rk)}}{2},$$

$$y=\frac{-2k+\sqrt{(2r^2+2rk)}-\sqrt{(2r^2-2rk)}}{2},$$

the center of the circle inscribed in ABC.

Solving the relations (1) and (2), we get

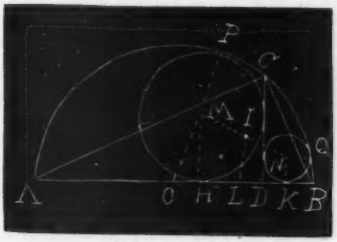
$$x_1=y_1=-k-r+\sqrt{(2r^2+2rk)},$$
$$x_{11}=-y_{11}=-r+k+\sqrt{(2r^2-2rk)}.$$

The x's represent the radii of the three circles, and satisfy the required conditions $x=\frac{x_1+x_{11}}{2}$, and the centers are in a straight line since they satisfy the condition

$$\frac{y-y_1}{x-x_1}=\frac{y_{11}-y_1}{x_{11}-x_1}.$$

II. Solution by A. M. HARDING, A. M., University of Arkansas, and the PROPOSER.

Let M be the center of circle touching arc AC, sides AD and DC, and let H be the point where it touches AD. Let N be center of circle touching arc CB, sides CD and DB, and K the point where it touches DB. Also let I be center of circle inscribed in triangle ABC. $MH=HD$ since $\angle MDH=45°$, and similarly, $NK=DK$. Therefore $HM+NK=HK$.



Again, by a well known theorem, $BC=BH$. Hence $\angle HCB=\angle CHB=\angle CAH+\angle ACH$, but $\angle BCD=\angle CAH$; therefore, $\angle HCD=\angle HCA$. Hence $\angle DAI+\angle ACH=45°$. Again, $\angle ACI=45°$, therefore $\angle DAI=\angle HCI$. Hence, the four points A, C, I, and H are concyclic. Hence, $\angle IHD=\angle ACI=45°$. In a similar manner it can be shown that $\angle IKD=45°$.

Now let L be the point where the circle inscribed in triangle ABC touches AB; then since $\angle ILH=90°$, and $\angle IHL=\angle IKL=45°$, line $IL=HL$ and $IL=LK$; but IL is the radius of circle inscribed in triangle ABC. Therefore $IL=\frac{1}{2}HK=\frac{1}{2}(MH+NK)$ by above.

Again, since L is the mid-point of HK, and IL, which$=\frac{1}{2}(MH+NK)$, is also parallel to MH and NK, it is easily seen that I is the mid-point of the line MN.

Also solved similarly by J. Scheffer.

CALCULUS.

318. Proposed by JOHN C. GREGG, Greencastle, Ind.

A thread is wound spirally n times around a cone, the radius of whose base is r, and slant height h, the turns being at uniform distance apart. If the thread is kept taut, what will be the length of the trace of its end on a horizontal plane?

I. Solution by B. F. FINKEL, Ph. D., Drury College.

A solution of this problem, by Henry Gunder, of Findlay (Ohio) College, was published in Vol. 9, page 199, of the *School Visitor*, published by John S. Royer. Professor Gunder's solution is wholly incorrect. So far as we know, no correct solution of the problem has ever been published. The problem as stated is indeterminate; for the reason that a string stretched taut on the surface of a cone cannot be in equilibrium unless held in position by the friction of the surface of the cone, by an adhesive substance, or else by pegs driven into the cone normal to the plane formed by the tangent to the string and the corresponding element of the cone. In the case of friction holding the string in position, it could not be wound on the cone nor unwound from it by merely taking hold of one end of the string and stretching it taut. Were a surface formed by conceiving a normal to the tangent of the string and the element of the cone, to move along the string, then the string could be wound on the cone by taking hold of one end of it and keeping it taut. The string would come in contact with this surface above the surface of the cone, and as the winding proceeds, the string would slip on this surface until it came in contact with the surface of the cone. The string would thus lie in the groove formed by the surface of the cone and the surface generated by a normal moving along the curve, formed by the proposed position of the string.

Since the unwinding as required in the problem is impossible, the problem is impossible of solution. If we suppose the string to adhere slightly to the cone, there are a number of ways which might be proposed to unwind the string. Of these, two would naturally suggest themselves. We might suppose, first, the string unwound in such a manner that the tangent to the projection of the string on the plane of the base and the unwound portion of the string are in the same plane; and second, that the unwound portion of the string and the altitude of the cone lie in the same plane. Both of these assumptions lead to very complicated computations. We shall now show, by a different method, what Professor Gunder obtained as a result, and also indicate a method of solution according to the two assumptions made above.

Let $C-AD'DBA$ be the cone, whose radius $OB=R$ and altitude $CO=h$; P, any point on the string, whose coördinates are (x, y, z); P', a consecutive point; $OP_1=\rho$, and $\angle BOP_1=\theta$. Draw the lines CPD, $CP'D'$, OD, OD', O_1P, and O_1K. P_1 is the projection of P; P_1' of P', and K_1 of K. P'_1P_1B is the projection of the string on the xy-plane; K_1P_1 is the projection of KP, and P'_1P_1 is the projection of IK and $P'P$.

The equation of the curve of the string is obtained by setting up a relation between $(z, \rho, \theta, n, R, h)$ or $(x, y, z, \theta, n, R, h)$.

In the similar triangles DP_1P and DOC, we have $R-\rho : R=z : h$, or

$$\rho=\frac{R(h-z)}{h} \ldots (1).$$

Also, since the string makes n equidistant turns, we have arc DB : $2\pi Rn = z:h$, or $R\theta : 2\pi nR = z:h$, or

$$z = \frac{\theta h}{2\pi n} \ldots (2); \quad \therefore \rho = \frac{R}{2\pi n}(2\pi n - \theta) \ldots (3).$$

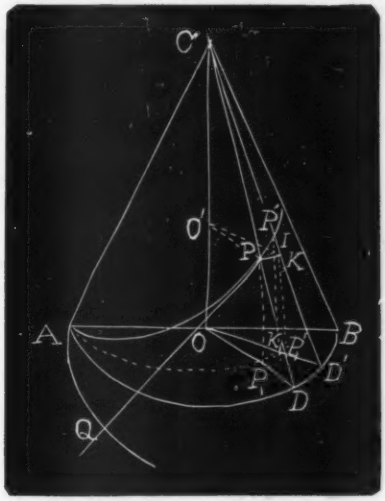
the equation of the projection of the curve of the string on the xy-plane.

Also, $x = \rho \cos\theta = \frac{R}{h}(h-z)\cos\theta$

$= \frac{R}{2\pi n}(2\pi n - \theta)\cos\theta \ldots (4)$; $y = \rho \sin\theta$

$= \frac{R(h-z)}{h}\sin\theta = \frac{R}{2\pi n}(2\pi n - \theta)\sin\theta$

$\ldots (5)$, $z = \frac{\theta h}{2\pi n}$ are the parametric equations of the curve of the string. The equations of the tangent to the string are:



$$\frac{\alpha - x}{\frac{dx}{ds}} = \frac{\beta - y}{\frac{dy}{ds}} = \frac{\gamma - z}{\frac{dz}{ds}}; \text{ or}$$

$$\frac{\alpha - x}{-\frac{R}{2\pi n}[\cos\theta + (2\pi n - \theta)\sin\theta]}$$

$$= \frac{\beta - y}{-\frac{R}{2\pi n}[\sin\theta - (2\pi n - \theta)\cos\theta]} = \frac{\gamma - z}{\frac{h}{2\pi n}} \ldots (6).$$

The coördinates of the point of intersection of the tangent with the xy-plane are found by letting $\gamma = 0$ in (6). Hence,

$$\alpha = \frac{R}{2\pi n}[(2\pi n - \theta)(\cos\theta + \theta\sin\theta) + \theta\cos\theta] \ldots (7), \text{ and}$$

$$\beta = \frac{R}{2\pi n}[(2\pi n - \theta)(\sin\theta - \theta\cos\theta) + \theta\sin\theta] \ldots (8).$$

The length of the curve described by the point of intersection of the tangent with the xy-plane is

$$s=\int \sqrt{[d\alpha^2+d\beta^2]}=\frac{R}{2\pi n}\int_0^{2\pi n}\theta\sqrt{[4+(2\pi n-\theta)^2]}\,d\theta$$

$$=\frac{4R}{3\pi n}+\frac{2}{3}R\left(\pi n-\frac{2}{\pi n}\right)\sqrt{[1+\pi^2n^2]}+2R\log[\pi n+\sqrt{(1+\pi^2n^2)}].$$

This is the result obtained by Professor Gunder in the solution referred to above, and cannot be correct unless the string, in unwinding, is "taken up" at the same time; for the portion of the string unwound at any time is longer than the tangent to the string at the point of contact with the cone. The length of a portion of the string is

$$l=\int \sqrt{[dx^2+dy^2+dz^2]}=\frac{1}{2\pi n}\int_0^{\theta}\sqrt{[h^2+R^2(2\pi n-\theta)^2]}\,d\theta$$

$$=\frac{1}{2\pi n}\left[-\frac{R}{2}(2\pi n-\theta)\sqrt{\frac{h^2+R^2}{R^2}+(2\pi n-\theta)^2}-R\left(\frac{h^2+R^2}{R^2}\right)\log\left[(2\pi n-\theta)\right.\right.$$

$$\left.+\sqrt{\frac{h^2+R^2}{R^2}+(2\pi n-\theta)^2}\right]+\pi n R\sqrt{\frac{h^2+R^2}{R^2}+4\pi^2n^2}$$

$$\left.+R\left(\frac{h^2+R^2}{R^2}\right)\log\left(2\pi n+\sqrt{\frac{h^2+R^2}{R^2}+4\pi^2n^2}\right)\right]$$

$$=\frac{1}{2\pi n}\left[\pi n\sqrt{[h^2+R^2+4\pi^2n^2R^2]}-\tfrac{1}{2}\sqrt{[2\pi n-\theta]}\sqrt{[h^2+R^2+(2\pi n-\theta)^2R^2]}\right.$$

$$\left.+R\left(\frac{h^2+R^2}{R^2}\right)\log\left(\frac{2\pi nR+\sqrt{[h^2+R^2+4\pi^2n^2R^2]}}{[2\pi n-\theta]R+\sqrt{[h^2+R^2+R^2(2\pi n-\theta]^2}}\right)\right].$$

The length of the corresponding tangent is

$$s=\sqrt{[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}=\frac{\theta}{2\pi n}\sqrt{[h^2+R^2(2\pi n-\theta)^2]}.$$

A comparison of these two expressions shows that the length of the unwound portion of the string is longer than the tangent to the point of contact. This fact also shows that the string cannot be unwound from the cone in the manner required by the problem unless it is held in contact with the cone by means of some force, as, for example, an adhesive substance.

Since the string must be held in position on the cone, we may choose any number of ways of unwinding it.

Of these ways, we have already mentioned two. First, suppose the string is unwound in such a way that the projection of the string on the xy-plane is the tangent to the curve $P_1'P_1B$. The equation of the trace of the string when unwound from the cone, by keeping it taut, is

$$OQ=[P_1Q^2+OP_1^2-2P_1Q.OP\cos\angle OP_1Q]^{\frac{1}{2}},$$

where $P_1Q=\sqrt{[PQ^2-P_1P^2]}=\sqrt{[s^2-z^2]}$, s being the unwound portion of the string, and $\angle QP_1O=\tan^{-1}[\rho\, d\theta/d\rho]$; or

$$OQ=\sqrt{s^2-z^2+\rho^2-\frac{2\rho\sqrt{[s^2-z^2]}}{[1+(2\pi n-\theta)^2]^{\frac{1}{2}}}}.$$

From this the length of the path may be found by approximate methods.

Second, suppose that the string is unwound so that the unwound part lies always in a plane with the axis of the cone. In this case the equation of the trace is $OQ=\sqrt{[s^2-z^2]}+\rho$, where s is the unwound portion of the string, and $\rho=OP_1$. The length of the trace in this case is

$$l=\int_0^{2\pi n}\{[(sds-zdz)(s^2-z^2)^{-\frac{1}{2}}+d\rho]^2+[\sqrt{(s^2-z^2)}+\rho]^2 d\theta^2\}^{\frac{1}{2}}.$$

A solution, by approximate quadrature, may be completed in a manner similar to that indicated in my solution of problem 309, pages 107-110, Vol. XIII of MONTHLY.

II. Solution by PROFESSOR F. L. GRIFFIN, Reed College, Portland, Oregon.

I. *Preliminary remark.* The problem is impossible in the absence of friction; since a stretched string can be in equilibrium on a "smooth" surface only along a geodesic line, and no geodesic line has the proposed location. Choosing the axis of the cone as the Z-axis, and the vertex as the origin, the equation of the surface is

$$f(x, y, z)\equiv x^2+y^2-z^2\tan^2\alpha=0,$$

where α is the half-angle of the cone, or $\arctan(r/h)$. The differential equations of a geodesic are

$$\frac{d^2x}{ds^2}=\nu\frac{\partial f}{\partial x}=2\nu x,\quad \frac{d^2y}{ds^2}=\nu\frac{\partial f}{\partial y}=2\nu y,\quad \frac{d^2z}{ds^2}=\nu\frac{\partial f}{dz}=-2\nu z\tan^2\alpha,$$

where s denotes the length of the arc and

$$r=\sqrt{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2}\Big/\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}.$$

Combining the first pair of equations gives an integral, $x\frac{dy}{ds}-y\frac{dx}{ds}=c$, constant. But since for any point (x, y, z) on the cone, $\sqrt{[x^2+y^2]}=z\tan\alpha$, we have $x=z\tan\alpha\cos\phi$, $y=z\tan\alpha\sin\phi$, where ϕ denotes the polar angle of the projection of (x, y, z) on the horizontal reference plane. Then

$$\frac{dx}{ds}=\tan\alpha\left[\frac{dz}{ds}\cos\phi-z\sin\phi\frac{d\phi}{ds}\right],\quad \frac{dy}{ds}=\tan\alpha\left[\frac{dz}{ds}\sin\phi+z\cos\phi\frac{d\phi}{ds}\right],$$

whence the integral becomes $z^2\tan^2\alpha(d\phi/ds)=c$. (If $c=0$, then either $z=0$ permanently or else $\phi=$constant, showing,—as is physically obvious,—that a thread cannot be thus wound up beginning at the vertex). Further,

$$1=\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=\tan^2\alpha\left[\left(\frac{dz}{ds}\right)^2+z^2\left(\frac{d\phi}{ds}\right)^2\right]+\left(\frac{dz}{ds}\right)^2$$

$$=\left[\left(\frac{dz}{d\phi}\right)^2\sec^2\alpha+z^2\tan^2\alpha\right]c^2/z^4\tan^4\alpha.$$

Separating variables, we have

$$\frac{dz}{z\sqrt{[z^2-c^2\cot^2\alpha]}}=-\frac{\sin\alpha\tan\alpha}{c}d\phi,$$

whence, k being constant. $z=c\cot\alpha\sec(k c\cot\alpha-\phi\sin\alpha)$.

Evidently, for small enough values of $\sin\alpha$ the geodesic may wind several times around the cone before receding to infinity at a certain finite value of ϕ; but even these few turns are never at uniform distance apart. For, if adding 2π to ϕ merely adds some constant to z, it follows that $(dz/d\phi)$ is the same at $\phi+2\pi$ as at ϕ, since

$$\frac{z(\phi+\Delta\phi+2\pi)-z(\phi+2\pi)}{\Delta\phi}\equiv\frac{z(\phi+\Delta\phi)-z(\phi)}{\Delta\phi};$$

but clearly the period of $dz/d\phi$ is $2\pi/\sin\alpha$.

II. But if we waive this question, and suppose the surface of the cone

sufficiently rough to hold the thread in the proposed position; or if we think of the equivalent purely geometric problem, the required length, σ, of the trace of the end-point (ξ, η, ζ), may be found, or at least expressed as a definite integral, as follows:

The requirement $z(\phi+2\pi)-z(\phi)\equiv m$, constant, does not define z as a function of ϕ, but is most simply satisfied by $z=h-m\phi/2\pi$, or $\frac{dz}{d\phi}=-\frac{m}{2\pi}$. This gives, retaining z for brevity:

$$\frac{dx}{d\phi}=\tan\alpha\left[-\frac{m}{2\pi}\cos\phi-z\sin\phi\right],\quad \frac{d^2x}{d\phi^2}=\tan\alpha\left[\frac{m}{\pi}\sin\phi-z\cos\phi\right],$$

$$\frac{dy}{d\phi}=\tan\alpha\left[-\frac{m}{2\pi}\sin\phi+z\cos\phi\right],\quad \frac{d^2y}{d\phi^2}=\tan\alpha\left[-\frac{m}{\pi}\cos\phi-z\sin\phi\right],$$

$$\left(\frac{ds}{d\phi}\right)^2=z^2\tan^2\alpha+\frac{m^2}{4\pi^2}\sec^2\alpha,\quad \frac{d^2s}{d\phi^2}=z\tan^2\alpha\left(-\frac{m}{2\pi}\right)/\frac{ds}{d\phi}.$$

From $ds/d\phi$ it is easy to obtain s, the length of the curve on the cone to any point, or, in particular, to $\phi=2\pi n$, which gives the total length, l, of the string. Now we find from the equation of the tangent at any point (x, y, z) of this spiral curve:

$$\frac{\xi-x}{\frac{dx}{ds}}=\frac{\eta-y}{\frac{dy}{ds}}=\frac{\zeta-z}{\frac{dz}{ds}}=l-s=\text{length yet unwound.}$$

Hence, $\zeta=x+(l-s)\frac{dx}{d\phi}/\frac{ds}{d\phi}$, $\frac{d\xi}{d\phi}=(l-s)\,\frac{\frac{ds}{d\phi}\cdot\frac{d^2x}{d\phi^2}-\frac{dx}{d\phi}\cdot\frac{d^2s}{d\phi^2}}{\left(\frac{ds}{d\phi}\right)^2}$, and similarly for η, $d\eta/d\phi$. To get σ, the distance travelled by the projected point $(\xi, \eta, 0)$, we use $d\sigma^2=d\xi^2+d\eta^2$; thus

$$\left(\frac{d\sigma}{d\phi}\right)^2=(l-s)^2/\left(\frac{ds}{d\phi}\right)^4\tan^2\alpha\left[\left(\frac{ds}{d\phi}\right)^2\left(\frac{m^2}{\pi^2}+z^2\right)\right.$$

$$\left.+\frac{ds}{d\phi}\frac{d^2s}{d\phi^2}\left(\frac{m}{\pi}z\right)+\left(\frac{d^2s}{d\phi^2}\right)^2\left(\frac{m^2}{4\pi^2}+z^2\right)\right].$$

Every term in the right member being a known function of ϕ, we denote the entire member by $F(\phi)$, and write

$$\sigma=\int_0^{2n\pi}\sqrt{[F(\phi)]}d\phi.$$

Various reductions are possible in the integrand, but they do not seem to lead to an expression of the indefinite integral.

II. *Remark.* On a "rough" cone the thread may be wound following other curves than geodesics; and with sufficient friction the turns may be wound at uniform distance apart. But even this is impossible if the end of the thread be required to remain in some horizontal plane; so that the proposed problem is impossible (even with friction) if by "trace" we are to understand a curve actually travelled by the end of the thread. Let us prove this.

At every instant the unwound taut thread must lie in the plane tangent to the cone at the point of contact, P; so likewise must the line PT tangent to the curve on the cone at P. But the direction of PT (or of the next succeeding "element of arc") is determined by the intersection of the conical surface with the plane which is determined by the taut thread and the instantaneous direction of motion of its end-point. Since the tangent line PT lies both in the latter plane and in the tangent plane, PT must coincide with the taut thread which lies in these two planes. Thus in winding the thread about the cone, the unwound portion must at every instant be tangent to the curve on the cone.

Now if the end-point (ξ, η, ς) is to remain constantly in a plane at any distance g from the equation of PT, the tangent at (x, y, z),

$$\frac{\xi-x}{\frac{dx}{ds}}=\frac{\eta-y}{\frac{dy}{ds}}=\frac{\varsigma-z}{\frac{dz}{ds}}=l-s, \text{ (the length yet unwound).}$$

Separating variables in the last equation, $dz/(q-z)=ds/(l-s)$, whence, p being constant, $p(q-z)=(l-s)$. From this, $p^2dz^2=ds^2=dx^2+dy^2+dz^2$, so that $(p^2-1)dz^2=dx^2+dy^2=\tan^2\alpha(dz^2+z^2d\phi^2)$, since as in (I) above, $r=\tan\alpha.z\cos\phi$, etc. Thus $dz^2(p^2-\sec^2\alpha)=z^2\tan^2\alpha.d\phi^2$, or

$$\frac{dz}{z}=\frac{\pm\tan\alpha}{\sqrt{[p^2-\sec^2\alpha]}}d\phi=\pm Kd\phi, \text{ say,}$$

the upper or lower sign being taken according as the winding proceeds away from or toward the vertex, and the constant K being evidently real for a real curve. The integral $z=A.e\pm K\phi$ shows both that the winding process cannot begin at the vertex, since z vanishes only if $A=0$ in which case $z\equiv 0$, and also that the turns are not at a uniform distance apart, since the addition of 2π to ϕ serves merely to multiply z by a constant factor.

III. Suppose now that the end-point (ξ, η, s) moves so that the turns on the cone ("rough," of course) are equidistant as proposed, and let us find the length of the curve traced by the projection $(\xi, \eta, 0)$.

The requirement of equidistant turns, viz., $z(\phi+2\pi)=z(\phi)+m$, where the constant m is negative if the turns approach the vertex, does not define z as a function of ϕ, but it can be most simply satisfied by taking $z=A+m\phi/2\pi$, A being any constant not zero. Then

$$dx=\tan\alpha\left[\frac{m}{2\pi}\cos\phi-z\sin\phi\right]d\phi,\quad dy=\tan\alpha\left[\frac{m}{2\pi}\sin\phi+z\cos\phi\right]d\phi,$$

where z is written for brevity. Then

$$ds^2=\left[\frac{m^2}{4\pi^2}\sec^2\alpha+\tan^2\alpha\left(A+\frac{m\phi}{2\pi}\right)^2\right]d\phi^2,$$

from which s can be obtained by a simple integration; and in particular the entire length l may be found in terms of n, α and the practically arbitrary A, m. Then to find the length, σ, of the path of $(\xi, \eta, 0)$ we have

$$\xi=x+(l-s)\frac{dx}{d\phi}\Big/\frac{ds}{d\phi},\quad \eta=y+(l-s)\frac{dy}{d\phi}\Big/\frac{ds}{d\phi};$$

$$\frac{d\xi}{d\phi}=(l-s)\cdot\frac{\frac{ds}{d\phi}\frac{d^2x}{d\phi^2}-\frac{d^2s}{d\phi^2}\frac{dx}{d\phi}}{\left(\frac{ds}{d\phi}\right)^2},\quad \frac{d\eta}{d\phi}=(l-s)\frac{\frac{ds}{d\phi}\cdot\frac{d^2y}{d\phi^2}-\frac{d^2s}{d\phi^2}\frac{dy}{d\phi}}{\left(\frac{ds}{d\phi}\right)^2};$$

$$\left(\frac{d\sigma}{d\phi}\right)^2=\left(\frac{d\xi}{d\phi}\right)^2+\left(\frac{d\eta}{d\phi}\right)^2=(l-s)^2(\ \ldots$$

as on bottom of page 107, except that the former m is here $-m$, etc., to former conclusion).

IV. A thread whose turns about a cone have been made uniformly distant by the help of friction may be *unwound* without the end-point retracing its former path; and as the unwound portion need not now be tangent to the curve on the cone, the end-point may travel in a horizontal plane. Let us find the length of its path, again taking arbitrarily $z=A+m\phi/2\pi$. The taut thread must still be in the tangent plane: $(\xi-x)x+(\eta-y)y+(q-z)(-z\tan^2\alpha)=0$; and $(\xi-x)^2+(\eta-y)^2+(s-z)^2=(l-s)^2$. Now let $\xi-x=(l-s)\cos a$, $\eta-y=(l-s)\cos b$, $q-z=(l-s)\cos c$, introducing the direction-cosines of the

string. Then $\cos^2 a+\cos^2 b=\sin^2 c$, and $\cos a.\cos\phi+\cos b\sin\phi=\operatorname{cosec}\tan\alpha$, (dividing x and y by $z\tan\alpha$). Combining the first of these equations with that obtained by squaring the second, we find $\cos a\sin\phi-\cos b\cos\phi=\pm\sqrt{[1-\cos^2 c\sec^2\alpha]}$. Thus

$$\cos a=(\xi-x)/(l-s)=\pm\sin\phi\sqrt{[1-\cos^2 c\sec^2\alpha]}+\operatorname{cosec}.\tan\alpha.\cos\phi;$$
$$\cos b=(\eta-y)/(l-s)=\mp\cos\phi\sqrt{[1-\cos^2 c\sec^2\alpha]}+\operatorname{cosec}.\tan\alpha.\sin\phi.$$

Since x, y, z, s, cosec are all known functions of ϕ, these equations furnish $d\xi$ and $d\eta$ in terms of ϕ and $d\phi$, and thus the problem of finding σ is reduced to a quadrature.

NOTES AND NEWS.

Mr. C. H. Forsyth of Michigan University has accepted the chair of mathematics in Eureka College, Eureka, Illinois. M.

Mr. C. A. Barnhart, assistant in the University of Illinois, has accepted the chair of Mathematics in Carthage College, Carthage, Illinois. M.

Dr. R. K. Morley, instructor in the University of Illinois, has accepted an assistant professorship of mathematics in Worcester Polytechnic Institute, Worcester, Mass. M.

Dr. G. F. McEwen, instructor in mathematics in the University of Illinois, has accepted a position in the Marine Biological station of the University of California. This station is located at Lajolla, Cal. M.

A circular, bearing the signature of about seventy prominent mathematicians, calls attention to the fact that during August of the present year Professor Felix Klein of Göttingen, Germany, will reach the fortieth anniversary of his appointment as professor of mathematics in the University of Erlangen. It is proposed to give him some token of the thanks and the good wishes of his fellow mathematicians in view of his great services to mathematical progress, and of his excellent work as a mathematical teacher. Attention is called especially to the great influence of the so-called *Erlanger Programm*, which was translated into English by Professor Haskell, and which was published in the *Bulletin of the New York Mathematical Society*, volume 2 (1893), page 215, under the title, "A Comparative Review of Recent Researches in Geometry." It may be added that Klein is the president of the International Commission on the Teaching of Mathematics, which is expected to report to the International Congress of Mathematicians at its fifth meeting, which is to be held at Cambridge, England, during the coming August. M.

During the Summer Session of the University of Illinois, June 17 to August 9, 1912, the following mathematical courses will be offered: Advanced algebra, plane trigonometry, analytic geometry, differential calculus, integral calculus, differential equations, and projective geometry. Graduate credit may be granted only for work in the last two of these courses. M.

It may interest mathematicians to learn of a valuable series of card-catalogue cards which the Library of Congress will have ready for distribution next July. This series is the beginning of a Dictionary Catalogue of all articles in the *Encyklopädie der Mathematischen Wissenschaften* and *Encyclopédie des Sciences Mathématiques.* Brown University has already supplied copy for the catalogue of all parts of these works which have been published. It will continue to furnish copy for further cards as the various parts of the encyclopædias appear. R. C. ARCHIBALD.

The latest number of the *Revue Semestrielle*, covering the period of six months from April to October, 1911, classifies the mathematical literature which appeared during this period under about 250 different headings, excluding the sub-headings represented by small letters of the Roman and the Greek alphabets. The largest number of references appear under the headings of biography and various considerations on the philosophy and the teaching of mathematics. The other headings under each of which there are more than twenty references to literature appearing during the given six months are as follows: Functions of real variables, series and infinite developments, electrodynamics, thermodynamics, light, surfaces in general and lines traced on a surface, theory of equations, functional equations, calculus of probability, systems and families of surfaces, elasticity, determinants, integral calculus, algebraic and circular functions, particular linear differential equations, indeterminate analysis of order higher than the first, plane and spherical curves, dynamics of solids and of material systems, rational hydrodynamics, and the history of mathematics in the twentieth century. The numbers of these references are useful to determine the fields of greatest present mathematical activity. Other facts must, however, be also considered. M.

BOOKS.

Non-Euclidian Geometry. A Critical and Historical Study of Its Development. By Roberto Bonola, Professor in the University of Pavia. Authorized English translation with additional appendices, by H. S. Carslaw, Professor in the University of Sydney, N. S. W., with an introduction by Federigo Enriques, Professor in the University of Bologna. 8vo. Red cloth. xii+263 pages. Price, $2.00. Chicago: The Open Court Publishing Co.

This is, as it purports to be, a critical and historical study of non-Euclidean geome-

try. The first chapter deals with attempts to prove Euclid's parallel postulate, beginning with the Greeks, Euclid, Proclus, etc., and ends with the investigations of Wallis. The second chapter treats of the forerunners of non-Euclidean geometry, beginning with Gerolamo Sacchiri and ends with Wachter and Thibaut. Thibaut is alleged to be responsible for the erroneous proof of the angle-sum, by starting at one corner of a triangle and running the side along in its own trace until the next vertex is reached, then turning the side through an angle until it coincides with next adjacent side, and so on. Chapters III and IV deal with the founders of non-Euclidean geometry, beginning with Gauss and Bolyai and ending with Battaglini and Beltrami. Chapter V. has to do with the later developments of non-Euclidean geometry. In the five appendices are treated in order, The fundamental principles of statics and Euclid's postulate, Clifford's parallels and surface and sketch of Clifford-Klein' problem; the non-Euclidean parallel construction and other allied constructions; the independence of projective geometry from Euclid's postulate; and the impossibility of proving Euclid's parallel postulate. The book gives a most satisfactory treatment of the subject and will be welcomed by all interested teachers of Geometry. The publishers have done their part well in presenting this translation to English speaking mathematicians. F.

An Elementary Treatise on Statics. By S. A. Loney, M. A., Professor of Mathematics at the Royal Holloway College (University of London). Sometime Fellow of Sydney Sussex College, Cambridge. Large 8vo. Red Cloth. viii+393 pages. Price, $4.00 net. Cambridge, Eng.: The University Press. G. P. Putnam's Sons, American Agents.

This book is intended to serve as a companion to the author's excellent treatise on *Dynamics of a Particle and of Rigid Bodies.* It covers most of the usual subjects of Statics and requires on the part of the student a knowledge of the calculus and of solid geometry. A large collection of interesting problems are placed at the end of each chapter and after many of the articles. The treatment of the various subjects is in keeping with all the other works which Professor Loney has written. There is one thing, however, which most, if not all of his works lack, and that is a good index. The typography and binding is all that could be desired. F.

Kimball's Commercial Arithmetic. Prepared for use in Normal, Commercial, and High Schools and for the Higher Grades of the Common Schools. By Gustavus S. Kimball, Author of *Business English, Word Book, Business Speller,* etc.. 8vo. Cloth and Leather back. viii+418 pages. New York: G. P. Putnam's Sons.

A very good book for the commercial student. All the ordinary business transactions, such as Banking, Stocks and Bonds, and Insurance, are quite fully treated.

As is the usual way in books of this kind, many useful short-cuts for rapid calculation are clearly explained and illustrated by numerous illustrative examples. Of course, short-cuts are of little value educationally, but are useful to the accountant. The book is well conceived and well arranged. The answers are bound in a separate volume. F.

The Modern Locomotive. By C. Edgar Allen, A. M. I. Mach. E.; A. M. I. E. E. Small 8vo. Cloth, ix+174 pages. Cambridge, Eng.: The University Press. G. P. Putnam's Sons, American Agents.

This little volume is one of the Cambridge Manuals of Science and Literature. Its object is to sketch the general "principles governing the design and working of a modern locomotive and to trace the broad lines of development from its comparatively simple predecessor of twenty-five or thirty years ago." Much attention is given to combustion, transfer of heat, steam production, super heating, compounding, feedwater heating, resistance and stability. It has many valuable suggestions for the engineer. F.

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SOME APPRECIATIVE REMARKS ON THE THEORY OF NUMBERS.

By DR. G. A. MILLER, University of Illinois.

In volume 10, 1903, of this journal the writer of the present note gave a brief list of appreciative remarks on the theory of groups with a view to furnishing an easy means to gain a knowledge of some important elements of this theory. The present brief list of appreciative remarks on the theory of numbers is intended to serve a similar purpose. It should be remembered that such extracts frequently require some modifications, but they direct attention to central truths of great value.

In the introduction to Reid's *Elements of the Theory of Algebraic Numbers*, 1910, Professor Hilbert asserts that "up to the present there is indeed no other science so highly praised by its devotees as is the theory of numbers." Some American students of mathematics might at first be inclined to attach little significance to these words in view of the fact that the number of devotees of this subject may be supposed to be very small. It should be observed that the number of those who study the theory of numbers, especially in Germany, is not insignificant. If it is remembered that Kummer's classes in this subject at the time of his greatest popularity in the University of Berlin, numbered at least 250,* and that classes of more than 100 are not uncommon now, in this leading German university, it is evident that conclusions as regards the number of devotees of this subject should not be based on conditions in our own universities.

In 1907 Minkowski began the preface of his book, entitled *Diophantische Approximation*, with the following words: "Integral numbers are the source (Urquell) of all mathematics. By this I understand not only the old view according to which the concept of continuity can also be deduced out of the consideration of discrete aggregates. I think much more about later results in using these words. The facts that the theory of the division of the circle dominates the theory of expotential functions and that the el-

* *Festschrift zur Feier des 100 Geburtstages*, Edward Kummers, 1910, p. 15.